REDUCTION OF NUMERICAL DISPERSION BY OPTIMIZING SECOND-ORDER CROSS PRODUCT DERIVATIVE TERM IN THE ADI-FDTD METHOD

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ABSTRACT: We propose a new adaptive alternating-direction implicit finite-difference time-domain (ADI-FDTD) to reduce the anisotropy in the numerical phase velocities. The proposed form has two anisotropic parameters on the first order terms of the time step and one parameter on the second order term of the time step. We introduce a new approach of reducing the error of the numerical phase velocity by controlling the three parameters. The numerical stability is discussed and compared with the previous approach of reducing the numerical dispersion.

Key words: FDTD; ADI-FDTD; CFL condition; truncation error; dispersion

1. INTRODUCTION

The alternating-direction implicit finite-difference time-domain (ADI-FDTD) method is an unconditionally stable scheme because the time step size is no longer bounded by the Courant-Friedrichs-Lewy (CFL) number [1, 2]. But the numerical dispersion error and a splitting error are the great disadvantages of the ADI-FDTD method when the time step increases [3].

Several papers have investigated the dispersion errors and anisotropic behaviors associated with the ADI-FDTD algorithm. Two parameters are properly introduced in the ADI-FDTD discretization in [4], thus leading to a minimization of the dispersion error at two arbitrary incident angles. Zheng introduced anisotropy parameters into the ADI-FDTD algorithm to reduce the numerical dispersion [5]. Zheng’s idea is an extension of the ADI-FDTD algorithm, similar to the artificial anisotropy introduced by Juntunen and Tsiboukis to reduce the numerical dispersion error in the conventional FDTD method [6]. The algorithm puts the two anisotropy parameters in the first order time derivative terms in space and equalizes the phase velocity at 0° and 90° with suitable variables to minimize the numerical phase velocity error.

In this article, we follow the anisotropy parameters of [4, 5] in the first order terms in time and add an equalization parameter to reduce the numerical anisotropy by adding the second order term of x and y derivatives in space. The proposed method is found to eliminate numerical anisotropy over a wider bandwidth. Numerical experiments are carried out to validate the proposed formulation and confirm the numerical stability.

2. FORMULATION OF THE ADI-FDTD METHOD FOR REDUCING THE NUMERICAL ANISOTROPY

The 2D Maxwell’s equations for the TE-z case can be written as the following matrix system:

\[ \frac{\partial}{\partial t} \vec{U} = A \vec{U} + B \vec{U} \]  

where \( \vec{U} = [E_x, E_y, H_z]^T \).

Here \( \varepsilon \) and \( \mu \) are the permittivity and permeability in free space. By substituting the time derivative of (1) with the corresponding difference operator, we can obtain the following Crank-Nicolson form:

\[ \left( I - \frac{\Delta t}{2} A - \frac{\Delta t}{2} B \right) \vec{U}^{n+1} = \left( I + \frac{\Delta t}{2} A + \frac{\Delta t}{2} B \right) \vec{U}^n, \]  

which is inefficient to solve it directly, as well known. The difference equation can be rewritten as,

\[ I - \frac{\Delta t}{2} A \left[ I - \frac{\Delta t}{2} B \right] \vec{U}^{n+1} = I + \frac{\Delta t}{2} A \left[ I + \frac{\Delta t}{2} B \right] \vec{U}^n + \frac{\Delta t^2}{4} A B (\vec{U}^{n+1} - \vec{U}^n) \]  

Truncating the second-order term of time-step size and introducing the intermediate (\( n + 1/2 \)) step can generate the ADI-FDTD formulation. The Peaceman-Ranchford approach can exactly change (3) to the following two equations [2],

\[ I - \frac{\Delta t}{2} A \left[ I - \frac{\Delta t}{2} B \right] \vec{U}^{n+1/2} = I + \frac{\Delta t}{2} A \left[ I + \frac{\Delta t}{2} B \right] \vec{U}^n + \frac{\Delta t^2}{8} A B (\vec{U}^{n+1} - \vec{U}^n) \]  

The difference equation can be rewritten as,

\[ I - \frac{\Delta t}{2} A - \frac{\Delta t^2}{8} A B \left[ I - \frac{\Delta t}{2} B \right] \vec{U}^{n+1/2} = I + \frac{\Delta t}{2} A - \frac{\Delta t^2}{8} A B \left[ I + \frac{\Delta t}{2} B \right] \vec{U}^n + \frac{\Delta t^2}{8} A B (\vec{U}^{n+1} - \vec{U}^n) \]  

\[ I - \frac{\Delta t}{2} B - \frac{\Delta t^2}{8} A B \left[ I - \frac{\Delta t}{2} A \right] \vec{U}^{n+1/2} = I + \frac{\Delta t}{2} B - \frac{\Delta t^2}{8} A B \left[ I + \frac{\Delta t}{2} A \right] \vec{U}^n + \frac{\Delta t^2}{8} A B (\vec{U}^{n+1} - \vec{U}^n). \]  

Ignoring the last terms of (5) is reduced into an efficient two-step formulation with the second-order cross derivative term(AB) included, just like in the ADI fashion.

\[ I - \frac{\Delta t}{2} A - \frac{\Delta t^2}{8} A B \left[ I - \frac{\Delta t}{2} B \right] \vec{U}^{n+1/2} = I + \frac{\Delta t}{2} B - \frac{\Delta t^2}{8} A B \left[ I + \frac{\Delta t}{2} A \right] \vec{U}^n \]  

Solving (6) generates lower numerical error than the ADI-FDTD formulation due to half size (\( \vec{U}^{n+1/2} \)) in the temporal variation of the truncation term in (5) compared to the full size (\( \vec{U}^{n+1} - \vec{U}^n \)) in the ADI-FDTD method. Moreover, we introduce
a degree of freedom by adding a parameter $\alpha$ in the square term of time step in such a way to control numerical performances.

$$\left[ 1 - \frac{\Delta t}{2}A - \alpha \Delta t^2 AB \right] \ddot{U}^{n+1/2} = \left[ 1 + \frac{\Delta t}{2}B - \alpha \Delta t^2 AB \right] \ddot{U}^n \quad (7a)$$

$$\left[ 1 - \frac{\Delta t}{2}B - \alpha \Delta t^2 AB \right] \ddot{U}^{n+1} = \left[ 1 + \frac{\Delta t}{2}A - \alpha \Delta t^2 AB \right] \ddot{U}^{n+1/2} \quad (7b)$$

$\alpha = 0$ and $1/8$ correspond to the ADI-FDTD method and the above formulation of (6), respectively. These forms of (7) are clearly different from the ADI-FDTD formulation by introducing the square term in time multiplied by cross derivatives in two-dimensional space. In order words, we can consider Eq. (7) as the perturbed form of the ADI-FDTD by adding the $\alpha \Delta t^2 AB$ terms to reduce the anisotropy of the numerical phase velocities. We will use the tensor parameters $\varepsilon_x$ and $\varepsilon_y$ into (7) in order to control the numerical phase velocity by taking the artificial anisotropy parameters, as in [5] and [6]. Modification of the first iteration (7a) by adding these parameters produces the following first procedure.

First procedure

$$E_{x,i+1/1,j-1/1,j}^{n+1/2} - E_{x,i+1/1,j+1/1,j}^{n+1/2} = \frac{\Delta t}{2\mu} \left[ H_{x,i+1/1,j+1/1,j}^{n+1/2} - H_{x,i+1/1,j-1/1,j}^{n+1/2} \right]$$

$$E_{y,i+1/1,j-1/1,j}^{n+1/2} - E_{y,i+1/1,j+1/1,j}^{n+1/2} = \frac{\Delta t}{2\mu} \left[ H_{y,i+1/1,j+1/1,j}^{n+1/2} - H_{y,i+1/1,j-1/1,j}^{n+1/2} \right]$$

$$E_{z,i+1/1,j-1/1,j}^{n+1/2} - E_{z,i+1/1,j+1/1,j}^{n+1/2} = \frac{\Delta t}{2\mu} \left[ H_{z,i+1/1,j+1/1,j}^{n+1/2} - H_{z,i+1/1,j-1/1,j}^{n+1/2} \right]$$

$$H_{x,i+1/1,j-1/1,j}^{n+1/2} - H_{x,i+1/1,j+1/1,j}^{n+1/2} = \frac{\Delta t}{2\mu} \left[ E_{x,i+1/1,j+1/1,j}^{n+1/2} - E_{x,i+1/1,j-1/1,j}^{n+1/2} \right] - \frac{\alpha \Delta t^2 E_{x,i+1/1,j-1/1,j}^{n+1/2} - E_{x,i+1/1,j+1/1,j}^{n+1/2} + E_{x,i+1/1,j-1/1,j}^{n+1/2}}{\varepsilon_x \Delta y} \quad (8a)$$

$$H_{y,i+1/1,j-1/1,j}^{n+1/2} - H_{y,i+1/1,j+1/1,j}^{n+1/2} = \frac{\Delta t}{2\mu} \left[ E_{x,i+1/1,j+1/1,j}^{n+1/2} - E_{x,i+1/1,j-1/1,j}^{n+1/2} \right] - \frac{\alpha \Delta t^2 E_{y,i+1/1,j-1/1,j}^{n+1/2} - E_{y,i+1/1,j+1/1,j}^{n+1/2} + E_{y,i+1/1,j-1/1,j}^{n+1/2}}{\varepsilon_y \Delta x} \quad (8b)$$

$$H_{z,i+1/1,j-1/1,j}^{n+1/2} - H_{z,i+1/1,j+1/1,j}^{n+1/2} = \frac{\alpha \Delta t^2 E_{z,i+1/1,j-1/1,j}^{n+1/2} - E_{z,i+1/1,j+1/1,j}^{n+1/2} + E_{z,i+1/1,j-1/1,j}^{n+1/2}}{\varepsilon_x \Delta y \varepsilon_y \Delta x} \quad (8c)$$

$\Delta x$ and $\Delta y$ are the spatial step sizes. Similarly, the second procedure can be obtained from (7b). To solve these numerical system, we first solve the tri-diagonal matrix of $E_x$, (8b) and substitute it for the $E_y$ field of $E_x$ (8a).

The numerical stability of (7) will be studied using the approach in [7]. A 2-D TE-z wave is defined as follows:

$$\psi = \psi_{0}e^{j(tk_{x}x + k_{y}y)} \quad (9)$$

where $j = \sqrt{-1}$, $k_{x}$ and $k_{y}$ are wavenumbers along the $x$ and $y$ axes, and $\psi_{0}$ indicates the amplification factor. We can obtain the following two characteristic equations for the first and second subiterations of ADI time marches by inserting (9) into (8) and the corresponding second procedure of (8b) equation.

$$q_{x}\xi_{1} - 2r_{x}\xi_{1} + p = 0$$

$$p\xi_{2} - 2r_{x}\xi_{2} + q = 0 \quad (10)$$

where $\xi_{1}$ and $\xi_{2}$ are the individual amplification factors of the two sub-steps and $c = 1/\sqrt{\mu}$ is the wave velocity in free space. Moreover, $r_{x} = 1 + 2\alpha c^{2}W_{x}^{2}e_{x}, p = 1 + c^{2}W_{x}^{2}e_{x}$, and $q = 1 + c^{2}W_{x}^{2}e_{x}$ where $W_{x} = (\Delta t/\Delta x)\sin(k_{x}\Delta x/2)$ and $W_{y}$ = $(\Delta t/\Delta y)\sin(k_{y}\Delta y/2)$. Solving (10) simultaneously, we obtain the growth factors for the subiterations as follows:

$$\xi_{1} = \frac{r \pm \sqrt{q - r^{2}}}{q} \quad \xi_{2} = \frac{r \pm \sqrt{q - r^{2}}}{p} \quad (11)$$

Note that the growth factors have to be a magnitude of unity in order that the numerical scheme (7) will be unconditionally stable. So, we place an inequality constraint of $\alpha$, the coefficient of the second order term:

$$e_{x}(1 - \frac{\sqrt{pq}}{q}) \leq \alpha \leq e_{x}(1 + \frac{\sqrt{pq}}{q}) \quad (12)$$

The domain of $\alpha$ is determined by meshing densities, CFL number, propagation direction, and anisotropy parameters, $e_{x}$ and $e_{y}$. It is important to determine the area of $\alpha$ of (12), independently of wave propagation angle. If we set $X = \sin(k_{x}\Delta x/2)$ and $Y = \sin(k_{y}\Delta y/2)$, then the right hand boundary of (12) is a function of two parameters, $X$ and $Y$ of $0 \leq X \leq 1$ and $0 \leq Y \leq 1$ as follows:

$$- 1 + \sqrt{(1 + C_{1}X)(1 + C_{2}Y)/C_{3}X^{2}Y^{2}} \quad (13)$$

for the related constants $C_{1}$, $C_{2}$, and $C_{3}$ larger than 0.

It is a negative function with respect to $X$ and $Y$, which is on the domain $0 \leq X \leq 1$ and $0 \leq Y \leq 1$. Being a monotonically decreasing function for each parameter, it has a minimum value at the point $(X = 1, Y = 1)$ and this fact is always true for the variables, $C_{1}$, $C_{2}$, and $C_{3}$. Similarly, it can be proved that the left hand boundary of (13) is an increasing function with respect to $X$ and $Y$. Thus, it has a maximum value at the point $(1, 1)$. We redefine $\alpha_{L}$ and $\alpha_{R}$ as such maximum and minimum values. If we put $(X, Y) = (1, 1)$ in (13), the removed form of dependence of wave propagation angle can be defined as:

$$(\alpha_{L} = \frac{1}{\sqrt{C_{3}}} \sqrt{(1 + C_{1})(1 + C_{2})}) \frac{2c^{4}r^{4}}{e_{x}c^{2}X^{2}} \quad \leq \alpha \leq \frac{1}{\sqrt{C_{3}}} \sqrt{(1 + C_{1})(1 + C_{2})}) \frac{2c^{4}r^{4}}{e_{x}c^{2}X^{2}} \quad (14)$$

From the unity complete update condition of $|\xi| = |\xi_{1}| = 1$, we can obtain the governing equation of numerical dispersion:

$$\cos^{2}(\omega_{L}/2) = \frac{(2ac^{4}W_{x}^{2}/e_{x} + 1)^{2}}{(c^{2}W_{x}^{2}e_{x} + 1)(c^{2}W_{x}^{2}e_{x} + 1)} \quad (15)$$

Let us define the ratio of the Yee cell, $Z = \Delta x/\Delta y$, and the spatial resolution, $R = \lambda/\Delta x^{2} + \Delta y^{2}$, where $\lambda$ is the wavelength in vacuum. Denote $A = c^{2}/e_{x} = \lambda \sqrt{\mu}$ will be the normalized phase velocity and $\lambda$ and $c^{2}$ are the numerical wavelength and propagation velocity in the grid, respectively. Including a CFL number $S$, then we obtain:

$$\frac{\omega_{L}}{2} = \frac{\pi c \Delta t}{\lambda} = \frac{\pi S N_{L} e_{x}}{R \sqrt{1 + Z^{2}e_{x} + e_{y}^{2}}} \quad (16)$$
Relation into (15), then we have the dimensionless numerical dispersion

\[ \frac{\kappa \Delta y}{2} = \frac{\pi}{AR \sqrt{1 + Z^2}}. \]  

(17)

where \( \kappa \) is the numerical wavenumber. If we substitute (16)–(17) into (15), then we have the dimensionless numerical dispersion relation

\[ \left( \frac{S^2 e_s}{e_s + e_c} \sin^2(\eta Z \cos \phi) + 1 \right) \left( \frac{S^2 Z e_s}{e_s + e_c} \sin^2(\eta \sin \phi) + 1 \right) \cos(A S \eta \sqrt{e_s / e_c}) \]

\[ = \left( \frac{2aS^2 e_s}{(e_s + e_c)} \sin^2(\eta Z \cos \phi) \sin^2(\eta \sin \phi) + 1 \right)^2 \]  

(18)

where \( \eta = \pi AR \sqrt{1 + Z^2} \). Putting \( \phi = 0^\circ \) and \( \phi = 90^\circ \) in (18) respectively, we obtain

\[ e_s = a^2 + b^2 \left[ \tan^{-1} \left( \frac{abS}{\sqrt{a^2 + b^2}} \right) \right]^2 \]  

(19)

and

\[ e_r = \frac{a^2 + b^2 + a b S}{A S \eta \sqrt{e_c / e_s}} \left[ \tan^{-1} \left( \frac{abS}{\sqrt{a^2 + b^2}} \right) \right]^2 \]  

(20)

where \( b = \sin(\eta Z) \) and \( a = \sin(\eta) \). The above derivation procedure is well explained in [6] for the \( \alpha = 0 \) case. Anisotropy and equalization parameters are required to find the desired numerical results. Let us define \( \alpha_0 \) as the optimized value equalizing the phase velocity at \( \phi = 45^\circ \) and those of \( \phi = 0^\circ \) and \( 90^\circ \) for \( A = 1 \) in (18).

\[ \alpha_0 = \left( \frac{\frac{S^2 e_s}{e_s + e_c} \sin^2 \left( \frac{\eta Z}{2} \right) + 1}{\frac{2aS^2 e_s}{(e_s + e_c)} \sin^2(\eta Z \cos \phi) \sin^2(\eta \sin \phi) + 1} \right)^2 \cos \left( \frac{S Z e_s}{e_s + e_c} \right) \left( \frac{\pi}{2 R} \right) - 1 \]  

(21)

We used the definition terms and rules of [5, 6] in the above derivation procedure except for \( \alpha \) parameter term. We should check whether the value of \( \alpha_0 \) satisfies the stability condition of (14) or not for the chosen numerical parameters. Figure 1 shows \( \alpha_0 \) and \( \alpha_\theta \) versus \( Z \) for two numerical conditions. One is the unstable \( \alpha_0 \) greater than \( \alpha_\theta \) and the other is the stable \( \alpha_0 \) bounded between \( \alpha_0 \) and \( \alpha_\theta \) in (14). We will treat numerical experiments to reduce the anisotropy for two cases.

First we will consider an \( \alpha_\theta < \alpha_0 \) case of \( Z = 5, S = 5 \) and \( R = 20 \) (Fig. 2). Instead of the unstable \( \alpha_0 \) for the chosen numerical conditions, \( \alpha_\theta = 0.1337 \) found by (14) are chosen and the numerical phase velocity is plotted as a solid line with the numerical phase velocities of A-ADI FDTD as a dotted line for comparison [6]. The introduction of \( \alpha \) terms can reduce the anisotropic error, compared to the A-ADI-FDTD method.

Now we will redefine \( e_s \) and \( e_c \) to slow down the phase velocity by half of the phase velocity at \( \phi = 45^\circ \). We can obtain maximum \( A \), say \( A_{\text{max}} \) from (18) for the above \( \alpha_\theta \), \( e_s \), and \( e_c \) assuming \( Z = 1 \). The equation of \( A_{\text{max}} \) is an implicit form as follows:

\[ \left( \frac{S^2}{1 + Z^2} \sin^2 \left( \frac{\pi}{2 R A_{\text{max}}} \right) + 1 \right) \cos \left( \frac{\pi S}{2 R} \sqrt{e_s / e_c} \right) = \frac{aS^2 e_s}{2} \sin^2 \left( \frac{\pi}{2 R A_{\text{max}}} \right) - 1 \]  

(22)

Figure 1 Comparison of \( \alpha \) and \( \alpha_0 \) while varying \( Z \). (a) value of \( \alpha = \alpha_\theta \) and \( \alpha_0 \) while varying \( Z \). (S=5, R=20) (b) value of \( \alpha = \alpha_0 \) and \( \alpha_\theta \) while varying \( Z \). (S=1, R=10)

Figure 2 Plot of normalized phase velocity.
and it can be easily solved using Newton’s method. The maximum deviation $Q$ of $A$ from unity is defined by $(A_{\text{max}} - 1)$ [6] and we set $A = 1 - Q/2$. Using this $A_{\text{max}}$ and the finding procedure of $A$ in [6], we redefine $e_x$ and $e_y$ from (19)-(20). Figure 3 illustrates the results by redefinition of $e_x$ and $e_y$.

Figure 4 shows the dispersion errors of the proposed method for propagation directions of $0^\circ \sim 90^\circ$. It is observed that the anisotropy of the proposed formulation is more improved than that of the A-ADI-FDTD method due to the perturbed reaction of the second order term in time.

Secondly, the stable $\alpha_0$ of $Z = 1, S = 1, R = 10$ in Figure 1(b) will be investigated. After finding $\alpha_0$ in (18) and getting $e_x$ and $e_y$ from (19)-(20), the numerical phase velocity can be determined with the result of [6] for comparison as shown in Figure 5. In the second case, we can obtain almost isotropic propagation of the proposed method for all propagation directions. So we will omit the plot of the equivalent phase velocity errors of Figure 4.

### 3. ANALYSIS OF NUMERICAL ERRORS

In this section, the numerical error of the proposed formulation is obtained and compared with that of the A-ADI-FDTD method. In [6], Zheng clarified that the A-ADI FDTD method is more efficient than the traditional ADI FDTD method to reduce the numerical dispersion error. The 2-D TE-z problem illustrated inside Figure 6 consists of two 2-m-long parallel conducting plates in free space separated by a distance of 0.02 m, used in [8] and [9] for comparison of numerical errors. The numerical experiments were done with a 750-kHz raised cosine, which was held constant after reaching its maximum of 1 V, and cell sizes of 0.02 m along the $x$ and $y$ directions. The solid and dotted lines of $E_y$ fields along a horizontal cut through the parallel-plate structure in the steady state are obtained by the proposed method and Zheng’s A-ADI FDTD method. In the region between the plates $(4 \leq x \leq 6)$, the field amplitude is constant. Outside the plates, the field amplitude decays rapidly to zero. Taking $S = 3$ and $R = 17.67$, we can calculate $e_x, e_y$, and $\alpha$ by the same process in the previous chapter. Figure 6 shows that the numerical error of the proposed method is a little less than the A-ADI FDTD method’s due to the second order term in time. This implies that the proposed method can be
applied to numerical problems of large structures, especially involved with structures of fine geometry due to the dispersion error lower than the A-ADI FDTD method.

4. CONCLUSIONS
In this article, we propose a new adaptive ADI-FDTD method to reduce the numerical anisotropy of the numerical dispersion. The proposed form has two anisotropic parameters on the first order terms and one parameter on the second order term in the time step. Two cases of θ values by the chosen numerical parameters were studied to obtain the stable and optimum numerical results. The numerical errors of the proposed method were analyzed for two parallel conductors bounded by a PMC wall and compared with other results. It can be concluded that the proposed method can be efficiently applied to problems of large structures especially involved with structures of fine geometry due to the dispersion error lower than the A-ADI FDTD method.

REFERENCES

APLICATIONS OF RESONANT-TYPE METAMATERIAL TRANSMISSION LINES TO THE DESIGN OF ENHANCED BANDWIDTH COMPONENTS WITH COMPACT DIMENSIONS

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ABSTRACT: In this article, it is demonstrated that metamaterial transmission lines implemented by means of complementary split rings resonators are useful for the design of planar microwave components with small dimensions and broad operational bandwidths. These characteristics are achieved thanks to the small electrical size of complementary split rings resonators and to the controllability of the dispersion diagram in such artificial lines. The technique is illustrated by means of the design of a rat-race hybrid coupler. The performance and size of this metamaterial-based component is compared with those characteristics and dimensions of the classical (distributed) implementation. It is clearly demonstrated that by replacing the conventional transmission lines in the design with metamaterial transmission lines conveniently engineered, the functionality of the device is preserved over broader bandwidths and, additionally, significant size reduction is achieved. These characteristics and the absence of lumped elements in the design are of interest for applications in which cost and compatibility with fully planar technology are fundamental aspects. The most relevant contribution of the article is the demonstration of the compatibility between the resonant artificial lines and the design of broad band and high performance microwave components. © 2007 Wiley Periodicals, Inc.

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Key words: metamaterials; complementary SRRs; hybrid couplers

1. INTRODUCTION
Metamaterial transmission lines are one-dimensional propagating structures consisting on a host line loaded with reactive elements (see refs. 1-3 for a wide overview on this topic). The most outstanding property of these artificial lines is the controllability of their electrical characteristics (impedance and phase), which is superior than in conventional lines due to the presence of the loading elements. This allows us for the synthesis of artificial lines with extreme impedances and/or artificial lines where the electrical length is not directly related to the physical length (as occurs in conventional lines). This latter characteristic is a consequence of the controllable dispersion diagram (dispersion engineering) of such lines, and it has a fundamental implication: the required electrical lengths of the lines (which are determined by the specific applications) are achievable with structures whose dimensions are significantly smaller than those dimensions that result when using conventional lines. In many applications, the required phase shift (and impedance) of the lines can be satisfied by means of a single stage structure [4], this being the optimum solution for device miniaturization. At this point, we would like to mention that although the commonly accepted definition of metamaterial transmission line refers to an effectively homogeneous periodic structure (i.e. with period much smaller than guided wavelength) with controllable characteristics, we will be flexible and we will adopt under this term also those structures not satisfying the former requirement (homogeneity). The reason is that for microwave circuit design the aim is to achieve the required electrical characteristics at the design frequency, rather than to synthesize an effective medium.

Two main approaches have been proposed for the synthesis of metamaterial transmission lines: the dual transmission line model [5-7] and the resonant-type approach [8, 9]. In the former one, a host line is loaded with series connected capacitances and shunt inductances. Resonant type metamaterial transmission lines are implemented by loading a host line either with split rings resonators (combined with shunt inductances) [8] or with complementary split rings resonators (in combination with series capacitances) [9]. All these artificial transmission lines exhibit an unusual dispersion diagram, in which two transmission bands, separated by a frequency gap, do appear. Within the lower frequency band, the loading elements are dominant and wave propagation is backward (or left handed). Namely, the phase and group velocities are antiparallel and, assuming that the energy flows from left to right (positive direction), the phase constant, β, is negative. Left handed