The Simplest V-Cycle Fast Adaptive Composite Grid ADI-FDTD Method for Two-Dimensional Electromagnetic Simulations

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Abstract—An alternating-direction-implicit finite-difference time-domain (ADI-FDTD) method based on the fast-adaptive-composite grid ADI (FAC-ADI) scheme is introduced to reduce the truncation error generated when ADI-FDTD is derived from the Crank-Nicolson (CN) ADI scheme, and its efficiency is investigated. The relaxation equation and residual equation for FAC-FDTD are constructed by defining the error vector between the traditional ADI and CN FDTD formulas, and the iterative formulas are defined using the iterative schemes. A comparison of the computational efficiency between the FAC-ADI and ADI-FDTD is presented.

Index Terms—Fast-adaptive-composite (FAC) grid method, alternating-direction-implicit finite-difference time-domain (ADI-FDTD), iterative method, Maxwell’s equations.

I. INTRODUCTION

The finite-difference time-domain (FDTD) method is widely used to solve electromagnetic problems due to its low computational complexity and easy implementation [1]. However, some problems require small grid size with respect to wavelength that brings out small time step, as required by the Courant-Friedrichs-Lewy (CFL) stability condition for the FDTD method. The ADI-FDTD method is introduced to overcome the constraint of such small size in time step [2]. Although the ADI-FDTD method produces only a small computational requirement by its tridiagonal feature compared to another implicit method, i.e., Crank-Nicolson (CN) FDTD, it generates large numerical dispersion error due to its truncation term in its two-step factorization [3]. To reduce the splitting error, an iterative method has been proposed in [4], in which the traditional ADI-FDTD method is considered as a special case of the general iterative method. To improve the convergence rate, this iterative method incorporates the multigrid method of two computational domains, that is, fine grid and coarse grid which are defined on total simulation space [5]. Some numerical problems of ADI-FDTD require high resolution and accuracy in small local regions beyond what is required in the rest of the domain. The FDTD subgridding technique using multigrid method was proposed to increase resolution in regions of interest without increasing computational requirements [6]. Also, the 3-D hybrid ADI-FDTD/FDTD technique was proposed in [7].

In this letter, we consider the fast-adaptive-composite grid ADI (FAC-ADI) method which effectively treats such local demands. In order to improve the multigrid method’s efficiency, FAC-ADI restricts the fine grids to local subdomains where the error becomes large. FAC-ADI begins with the calculation of the ADI-FDTD on the global fine grid and uses the ADI subgridding method based on FAC method to reduce the truncation error between the ADI-FDTD and CN FDTD method. Note that the entire fine grid domain is only used to calculate the ADI-FDTD for data proceeding at the next time step.

II. FAC FORMULATION FOR 2-D ADI-FDTD METHOD

A. FAC Grid

For execution of the FAC scheme introduced in [5], four different uniform grids are needed, i.e., \( \Omega^h \), \( \Omega_{4c}^h \), \( \Omega^G^h \) and \( \Omega_{6c}^h \). The global fine grid \( \Omega^h \) is used for computing traditional ADI-FDTD equations. The local fine grid \( \Omega_{4c}^h \) is the finest grid for computing the relaxation equation; it covers only that part of the domain. The global coarse grid \( \Omega^G^h \) is the coarse grid used for computing the residual equation that covers the entire domain. The local coarse grid \( \Omega_{6c}^h \) appears when the interpolation method is used. When we use the FAC-ADI scheme, the solution is approximated on the composite grid as seen in Fig. 1, consisting of local fine grid, interface points and coarse grid. We define restriction function \( I_{6c}^h \) by full weighting of a fine-grid.
vector to the coarse grid, and the interpolation operator $I_{2h}^h$ is defined to be linear and satisfies the relation [8]

$$I_{2h}^h = c (I_h^h)^T$$

where $c$ is a constant and "$T$" denotes transpose. In the procedure of FAC, two equations have to be solved i.e., relaxation and residual equations. In the following section, we construct these equations for two dimensional electromagnetic simulations.

### B. Relaxation and Residual Equations for FAC-ADI

The 2-D Maxwell’s equations for the TE$_z$ wave in lossless medium can be written as the following matrix system divided into two matrices of $x$ and $y$

$$\frac{\partial}{\partial t} \mathbf{\tilde{u}} = \mathbf{A} \mathbf{\tilde{u}} + \mathbf{B} \mathbf{\tilde{u}}$$

(1)

where $\varepsilon$ and $\mu$ are the permittivity and permeability in free space and $\mathbf{\tilde{u}} = [E_x \ E_y \ H_z]^T$

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & \frac{\partial}{\partial y} \\ 0 & 0 & 0 \\ \frac{\partial}{\partial y} & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial x} & 0 \end{bmatrix}.$$  

Applying the CN scheme to (1) generates the following CN-FDTD equation:

$$\left( \mathbf{I} - \frac{\Delta t}{2} \mathbf{A} - \frac{\Delta t}{2} \mathbf{B} \right) \mathbf{\tilde{u}}^{n+1} = \left( \mathbf{I} + \frac{\Delta t}{2} \mathbf{A} + \frac{\Delta t}{2} \mathbf{B} \right) \mathbf{\tilde{u}}^n$$

(2)

where $\mathbf{I}$ denotes an identity matrix. The iterative ADI scheme is divided into two steps [4]

$$\left[ \mathbf{I} - \frac{\Delta t}{2} \mathbf{A} \right] \mathbf{\tilde{u}}^{mp} = \left[ \mathbf{I} + \frac{\Delta t}{2} \mathbf{B} \right] \mathbf{\tilde{u}}^n + \frac{\Delta t^2}{8} \mathbf{A} (\mathbf{\tilde{u}}^{n+1} - \mathbf{\tilde{u}}^n)$$

(3)

$$\left[ \mathbf{I} - \frac{\Delta t}{2} \mathbf{B} \right] \mathbf{\tilde{u}}^{n+1} = \left[ \mathbf{I} + \frac{\Delta t}{2} \mathbf{A} \right] \mathbf{\tilde{u}}^{mp} + \frac{\Delta t^2}{8} \mathbf{B} (\mathbf{\tilde{u}}^{n+1} - \mathbf{\tilde{u}}^n)$$

(4)

where $\mathbf{\tilde{u}}^{mp}$ is an intermediate solution. Note that if the initial condition of $\mathbf{\tilde{u}}^{n+1}$ is given as the value at the previous time step value, $\mathbf{\tilde{u}}^n$, (3)–(4) recover to the equations solved by the traditional ADI-FDTD scheme. The iterative scheme of (3)–(4) reduces the splitting error generated due to the difference between the traditional ADI-FDTD and CN FDTD method. This concept will be used to define the proposed method. From the iterative procedure of (3)–(4), Wang et al. [5] obtained the relaxation solution $\mathbf{\tilde{u}}^{n+1}$.

In this letter, we introduce a different approach to find the relaxation solution by starting calculation of the ADI scheme. It is similar to Wang’s iterative ADI scheme with the initial condition as the previous time step value. This method has a simple formula and does not have the necessity of calculations at computational interface points on $\Omega_h^0$. Moreover, this formula appears later in the residual equation defined on the coarse grid.

For each $(n+1)$th time step of the ADI-FDTD scheme, we first solve $\mathbf{\tilde{u}}^{n+1}_{\text{ADI}}$ on $\Omega_h$ from the ADI-FDTD method

$$\left[ \mathbf{I} - \frac{\Delta t}{2} \mathbf{B} \right] \left[ \mathbf{I} - \frac{\Delta t}{2} \mathbf{A} \right] \mathbf{\tilde{u}}^{n+1}_{\text{ADI}} = \left[ \mathbf{I} + \frac{\Delta t}{2} \mathbf{A} \right] \mathbf{\tilde{u}}^n$$

(5)
where $\tilde{u}_k^{n+1}$ is the solution in the previous time step. The linear system of (2) can be written by
\[
\begin{bmatrix}
I - \frac{\Delta t}{2} A & I - \frac{\Delta t}{2} B \\
I + \frac{\Delta t}{2} A & I + \frac{\Delta t}{2} B
\end{bmatrix}
\begin{bmatrix}
\tilde{u}_k^{n+1} \\
\tilde{v}_k^{n+1}
\end{bmatrix} =
\begin{bmatrix}
I - \frac{\Delta t}{2} A & I - \frac{\Delta t}{2} B \\
I + \frac{\Delta t}{2} A & I + \frac{\Delta t}{2} B
\end{bmatrix}
\begin{bmatrix}
\tilde{u}_k^n \\
\tilde{v}_k^n
\end{bmatrix} + \frac{\Delta t^2}{4} AB(\tilde{v}^{n+1} - \tilde{v}^n).
\]
\[\text{(6)}\]

We define a new vector $\tilde{e}_{\text{res}}^{n+1} = \tilde{e}_{\text{res}}^{n+1} - \tilde{e}_{\text{ADI}}^{n+1}$ describing the error difference between the ADI-FDTD and CN FDTD scheme. By elementary algebraic computation of (5) and (6), we obtain the following system:
\[
M \tilde{e}_{\text{res}}^{n+1} = N \tilde{e}_{\text{res}}^{n+1} + \tilde{r}^{n+1}
\]
\[\text{(7)}\]
where $M = (I - (\Delta t/2)A)(I - (\Delta t/2)B), N = (\Delta t^2/4)AB$ and $\tilde{r}^{n+1} = (\Delta t^2/4)AB(\tilde{u}_{\text{ADI}}^{n+1} - \tilde{u}^{n+1})$. Equation (7) can be regarded as a splitting scheme for solving the linear system $(M - N) \tilde{e}_{\text{res}}^{n+1} = \tilde{r}^{n+1}$ iteratively. Here, $M - N$ is a splitting of the matrix $(I - (\Delta t/2)A - (\Delta t/2)B)$. If we highlight the iterative nature of the equation, we have
\[
M_k \tilde{e}_{\text{res},k}^{n+1} = N_k \tilde{e}_{\text{res},k}^{n+1} + \tilde{r}_k^{n+1}
\]
\[\text{(8)}\]
where the subscript $k$ denotes the $k$th iterative solution. The convergence of the iterative method (8) is proved in [9]. Equation (8) is defined on $\Omega_k^{h,\text{loc}}$ and the value at the interface points is zero. To solve (8), we will use the two step formulation introduced in [5] which resembles the Peaceman–Rachford scheme. Equation (8) is divided into two steps
\[
\begin{bmatrix}
I - \frac{\Delta t}{2} A \\
I - \frac{\Delta t}{2} B
\end{bmatrix}
\begin{bmatrix}
\tilde{e}_{\text{mp}}^{n+1} \\
\tilde{e}_{\text{mp},k}^{n+1}
\end{bmatrix} = \frac{\Delta t^2}{8} AB(\tilde{u}_{\text{res},k}^{n+1} + \tilde{v}_{\text{res},k}^{n+1})
\]
\[\text{(9)}\]
\[
\begin{bmatrix}
I - \frac{\Delta t}{2} A \\
I - \frac{\Delta t}{2} B
\end{bmatrix}
\begin{bmatrix}
\tilde{v}_{\text{mp},k}^{n+1} \\
\tilde{v}_{\text{mp},k}^{n+1}
\end{bmatrix} = \frac{\Delta t^2}{8} AB(\tilde{u}_{\text{res},k}^{n+1} + \tilde{v}_{\text{res},k}^{n+1})
\]
\[\text{(10)}\]
where $\tilde{e}_{\text{mp}}^{n+1} = (\tilde{e}_{\text{mp},x}^{n+1}, \tilde{e}_{\text{mp},y}^{n+1}, \tilde{e}_{\text{mp},z}^{n+1})$ denotes an intermediate solution.

The relaxation solution can be obtained by solving (9)–(10)
\[
\begin{bmatrix}
I - \frac{\Delta t}{2} A \\
I - \frac{\Delta t}{2} B
\end{bmatrix}
\begin{bmatrix}
\tilde{u}_k^{n+1} \\
\tilde{v}_k^{n+1}
\end{bmatrix} = \tilde{e}_{\text{mp},k}^{n+1} + \tilde{r}_k^{n+1}
\]
\[\text{(11)}\]
Now we define the error vector $\tilde{e}_{\text{res}}^{n+1} = \tilde{u}_k^{n+1} - \tilde{u}_k^{n+1}$ for residual equation. The residual is given by
\[
\tilde{r} = \left(I - \frac{\Delta t}{2} A + \frac{\Delta t}{2} B\right) \tilde{u}_k^n - \left(I - \frac{\Delta t}{2} A - \frac{\Delta t}{2} B\right) \tilde{u}_{\text{ADI}}^{n+1}
\]
\[\text{(12)}\]
and transfer it to the coarse grid by full weighting operator which is an average of the fine-grid nearest neighbors [8]. According to (12) and the definition of $\tilde{r}$, the residual equation for the iterative ADI method on $\Omega^{h}$ satisfies the following:
\[
\left(I - \frac{\Delta t}{2} A - \frac{\Delta t}{2} B\right) \tilde{e}_{\text{res}}^{n+1} = \tilde{r}.
\]
\[\text{(13)}\]
Equation (13) equivalently can be converted to
\[
M \tilde{e}_{\text{res}}^{n+1} = N \tilde{e}_{\text{res}}^{n+1} + \tilde{r}
\]
\[\text{(14)}\]
and its iterative formula is
\[
M_k \tilde{e}_{\text{res},k}^{n+1} = N_k \tilde{e}_{\text{res},k}^{n+1} + \tilde{r}_k.
\]
\[\text{(15)}\]

Equations (15) and (8) are the same form except for remainder terms, and we can use the same iterative procedure of (9)–(10) to solve (15). We can solve (15) after calculating (8) by altering the remainder term of (8) to $\tilde{r}$. Since $\tilde{e}_{\text{res}}^{n}$ and $\tilde{e}_{\text{res}}^{n+1}$ are zero on the interface and boundary points, respectively, their calculations outside the boundary and interface are not required to solve (8) and (15). The splitting error between the ADI-FDTD and CN FDTD methods does not occur on the boundary of global and local domain and the effectiveness of the iterative scheme is realized only in the interior domain. The boundary condition is applied to solve only the ADI-FDTD method of (5) on the global domain. Finally, the solution of $(n + 1)$th time step is $\tilde{u}_k^{n+1} = \tilde{e}_{\text{res}}^{n+1} + \tilde{u}_k^{n+1}$.

C. Derivation of FAC ADI-FDTD Algorithm

In this letter, we apply the simplest V-cycle FAC-ADI for an example. The procedure of FAC-ADI first begins with the calculation of the ADI-FDTD scheme (5) on the fine grid $\Omega^{h}$. The further FAC algorithm appears as follows:

- Initialize $\tilde{u}_0^{h} = 0$ and $\tilde{u}_0^{n+1,h} = \tilde{u}_0^{h}$ for $n > 1$ obtained at the previous step.
- Relax on $\tilde{u}_0^{n+1,h}$ by solving (8) on the local fine grid $\Omega_{\text{loc}}^{h}$. 

Fig. 4. (a) Comparison between multgrid ADI and FAC-ADI method by varying the CFL numbers. (b) Relative error of $E_x$. The iterative number is 3.
coarse grids of full domain are applied and 40 × 5 local fine grids are over two plates on which high resolution and accuracy are required. The error grows large in the region containing the excitation source.

Fig. 3 shows the electric field $E_y$ distribution of FAC-ADI and the traditional ADI method along the x axis with different CFL numbers. In Fig. 3(a), outside the plates, the field amplitude decays more rapidly than that of the ADI-FDTD method with different CFL numbers of $\alpha = 3$ and 6. Fig. 3(b) illustrates how the effectiveness of iterative scheme is improved by varying the iterative number of relaxation and residual equations. It can be seen that the accuracy becomes better as the iteration number increases.

The FAC-ADI scheme can reduce the computation load compared to the multigrid ADI method [5] since the relaxation equation is defined and calculated on the local fine grid. Fig. 4(a) illustrates the efficiency between the simplest V-cycle multigrid ADI [5] and FAC-ADI method under the same gridsize ($\Delta x = 0.04$ m between the two parallel plates and $\Delta y = 0.02$ m, otherwise), and iterative number. The area over the fine grids is a third of the global domain, and therefore the memory and CPU time used with the FAC-ADI method are significantly decreased.

Fig. 4(b) shows the relative error of $E_y$ fields with different CFL numbers. The relative error is defined as

$$\frac{\sum_x |E_y - E_y^{ref}|}{\sum_x E_y^{ref}} \times 100$$

where $E_y$ is a measured field at $y = 2$ where the upper plate is located, and $E_y^{ref}$ is a reference field with value 1 at the feeding plate and 0 at the other area.

Fig. 5(a) shows the relative maximum error along x axis with CFL number 1 and 2. The relative maximum error is defined as

$$\frac{E_y - E_y^{ref}}{E_y^{ref}} \times 100$$

where $E_y$ is a measured field and $E_y^{ref}$ is a reference field obtained by computing the high number iterative ADI method on very fine mesh. The FAC-ADI relative error of the area near the boundary is large, but the FAC-ADI relative error of area with large absolute measured value is small. Fig. 5(b) shows that the FAC-ADI relative error of the area within 90% maximum measured value is almost the same as the multigrid ADI result.

IV. CONCLUSION

In this letter, the FAC-ADI scheme was introduced to improve the accuracy and reduce the splitting error between the traditional ADI-FDTD and CN ADI method. We first defined the relaxation equation of FAC-ADI on the local fine grid and its adaptive formula was derived by using the iterative ADI scheme. The residual equation was also defined on the coarse grid. Its governing formula is similar to the relaxation equation and this makes it efficient to get a solution. The numerical example of Fig. 3 illustrates that the FAC-ADI effectively reduces the splitting error. Figs. 4 and 5 graphs show a comparison with the multigrid ADI scheme.

III. NUMERICAL IMPLEMENTATIONS

For comparison of the numerical error with the ADI-FDTD method, two 2-m-long parallel conducting plates with a separation distance of 0.02 m in free space surrounded by perfect magnetic conductor (PMC) are investigated.

The coarse grid sizes are $\Delta x = 0.04$ m and $\Delta y = 0.04$ m in the x and y directions, respectively, and the fine grid sizes are half of the coarse grid as seen in Fig. 2. Numerical experiments were done with a 750-kHz raised cosine, which held constant after reaching its maximum of 1 V. In the region between the plates ($4 \text{ m} \leq x \leq 6 \text{ m}$), the field amplitude is constant. 25 × 10

- Compute fine-grid residual $r^{(b)}$ of (12) and transfer it to the local coarse grid $\Omega_k^{2h}$, i.e., $r^{2h} = f^{2h} - P_{k}^{2h} r^{(b)}$ at points underlying the local fine grid.
- Compute $r^{2h+1,2h}$ to the coarse-grid residual (15) on the global coarse grid $\Omega^{2h}$.
- Update the coarse-grid approximation $\tilde{u}_{k}^{n+1,2h} = \tilde{u}_{k}^{n+1,2h} + P_{k}^{2h} r^{2h+1,2h}$ at coarse-grid points of $\Omega^{2h}$.
- Interpolate the correction and update the approximation i.e., $u_{k}^{n+1,h} = u_{k}^{n+1,h} + I_{k}^{2h} r^{2h+1,2h}$ at the local fine grid $\Omega_{k}^{h}$ and interface points.
- Interpolate to the slave points from their interface neighbors i.e., $\tilde{u}_{k}^{n+1,j}(i + 1, j) \leftarrow \frac{u_{k}^{n+1,j}(i, j) + u_{k}^{n+1,j}(i + 2, j)}{2}$.
REFERENCES


